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THE BOUNDARY ELEMENT METHOD FOR THE DETERMINATION OF A TIME DEPENDENT HEAT SOURCE

A. FARCAS
BP Institute, University of Cambridge, Cambridge CB3 0EZ, UK
e-mail: adrian@bpi.cam.ac.uk

Abstract - This paper investigates the inverse problem of determining a heat source in the parabolic heat equation using the usual conditions of the direct problem and a supplementary condition, called overdetermination. In this problem, we take the heat source to be time-dependent only and the overdetermination is the temperature measurement taken at a single interior location. This measurement ensures that the inverse problem has a unique solution, but this solution is unstable, hence the problem is ill-posed. This instability is overcome using the Tikhonov regularization method with the the discrepancy principle for the choice of the regularization parameter. The boundary element method (BEM) is developed for solving numerically the inverse problem and numerical results for a test example are obtained and discussed.

1. INTRODUCTION

The inverse problem of determining a heat source function in the heat conduction equation has been considered in many theoretical papers, notably [1-5]. With the exception of [5], where the source is sought as a function of both space and time but it is additive or separable, in all the other studies the source has been sought as a function of space or time only. However, no numerical implementations have been attempted yet, under such a rigorous mathematical back-up. Although sufficient conditions for the unique solvability of the inverse problem were provided, the problem is still ill-posed since small errors, inherently present in any practical measurement, give rise to unbounded and highly oscillatory solutions. In this paper, in order to overcome this instability of the solution, the BEM combined with the Tikhonov regularization and the discrepancy principle for the choice of the regularization parameter is developed, see Section 3, for the numerical solution of the inverse problem formulated in Section 2. Section 4 discusses the numerical results for a test examples involving a time dependent heat source. The stability of the numerical solution is investigated by taking into account randomly perturbed noisy data. Finally, conclusions are presented in Section 5.

2. FORMULATION OF THE INVERSE PROBLEM

Let $T > 0$, $\alpha \in (0, 1)$ and $l > 0$ be fixed numbers and let us consider the problem in which the source $Q(x, t) = f(t)$ depends on time only. Hereafter we use the Sobolev functional space notation of [6].

We aim to find the temperature $u \in H^{2+\alpha,1+\alpha/2}([0,l] \times [0,T])$ and the heat source $f \in H^{\alpha/2}([0,T])$ which satisfy the heat conduction equation with a time-dependent source, namely

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(t), \qquad (x,t) \in (0,l) \times (0,T] \tag{1}$$

subject to the boundary conditions

$$u(0,t) = h_0(t), \quad u(l,t) = h_l(t), \qquad t \in [0,T], \tag{2}$$

the initial condition

$$u(x,0) = u_0(x), \qquad x \in [0,l] \tag{3}$$

and the overspecified condition

$$u(x_0,t) = \chi(t), \qquad t \in [0,T] \tag{4}$$

where $x_0 \in (0, l)$ is the interior location of a thermocouple recording the temperature measurement (4) and $h_0(t)$, $h_l(t)$, $u_0(x)$ and $\chi(t)$ are given functions.

We assume that the conditions (2)-(4) are consistent up to the first order, i.e.

$$u_0(0) = h_0(0), \quad u_0(l) = h_l(0), \quad u_0(x_0) = \chi(0) \tag{5}$$

$$h_0'(0) = u_0''(0) + \chi'(0) - u_0''(x_0), \quad h_l'(0) = u_0''(l) + \chi'(0) - u_0''(x_0) \tag{6}$$

so as to ensure the existence of a solution of the direct problem (1)-(3) if f is known. Then the unique solvability of the inverse problem (1)-(4) follows from the following theorem, see [2].

Theorem 1. *If* h_0, h_l, $\chi \in H^{1+\alpha/2}([0,T])$, $u_0 \in H^{2+\alpha}([0,l])$, *and the conditions (2)-(4) are consistent up to the first order, as given in (5) and (6), then the problem (1)-(4) has a unique solution* $(u, f) \in H^{2+\alpha,1+\alpha/2}([0,l] \times [0,T]) \times H^{\alpha/2}([0,T])$.

If instead of the condition (4) we specify a heat flux data, namely

$$-\frac{\partial u}{\partial x}(0,t) = q_0(t), \qquad t \in [0,T] \tag{7}$$

where $q_0(t)$ is a given function, then we have the following solvability theorem, see [7].

Theorem 2. *If* $h_0 = h_l \equiv 0$, $u_0 \in H_0^1(0,l)$ *and* $q_0 \in C^1(0,T)$ *then the inverse problem (1)-(3) and (7) has a unique solution* $(u, f) \in (H_0^1((0,l) \times (0,T)) \cap H^2((0,l) \times (0,T))) \times C(0,T)$.

Theorems 1 and 2 show that the inverse problems (1)-(4) and (1)-(3),(7), respectively, have unique solutions, but they are ill-posed since their solutions do not depend continuously on the input data. This can be seen for example by considering the problem (1)-(4) in which $u_0 \equiv 0$, $h_0(t) = h_l(t) = \chi(t) = \frac{1}{n}\sin(n^2 t)$ for $n \geq 1$. This problem has the unique solution $u_n(x,t) = \frac{1}{n}\sin(n^2 t)$, $f_n(t) = n\cos(n^2 t)$ and, as $n \to \infty$ the input data tends to zero, whilst the source $f_n(t)$ becomes oscillatory unbounded.

At this stage, we can remark that the inverse problem has a unique stable component of the solution, $u(x,t)$, and a unique, but unstable, component of the solution, $f(t)$. Therefore, in order to overcome this instability of the solution in the heat source component we employ the BEM combined with the Tikhonov regularization technique, as described in the next section.

3. THE BOUNDARY ELEMENT METHOD

By applying the Green's formula we can recast eqn. (1) in the integral form

$$\begin{aligned}\eta(x)u(x,t) = &\int_0^t \left[G(x,t,\xi,\tau)\frac{\partial u}{\partial n(\xi)}(\xi,\tau) - u(\xi,\tau)\frac{\partial G}{\partial n(\xi)}(x,t,\xi,\tau)\right] d\tau + \\ &+ \int_0^1 G(x,t,y,0)u(y,0)dy + \int_0^l \int_0^t G(x,t,y,\tau)Q(y,\tau)d\tau dy \\ &\text{for } (x,t) \in [0,l] \times (0,T], \quad \xi \in \{0,l\}\end{aligned} \tag{8}$$

where $\eta(0) = \eta(l) = 1/2$, $\eta(x) = 1$ for $x \in (0,l)$, n is the outward normal to the space boundary $\{0,l\} \times [0,T]$, i.e. $n(0) = -1$ and $n(l) = 1$, $Q(y,\tau) = f(\tau)$, and G is the fundamental solution of the one-dimensional heat equation, namely

$$G(x,t,y,\tau) = \frac{H(t-\tau)}{\sqrt{4\pi(t-\tau)}} exp\left(-\frac{(x-y)^2}{4(t-\tau)}\right) \tag{9}$$

where H is the Heaviside function.

Let us now discretize each of the boundaries $\{0\} \times [0,T]$ and $\{l\} \times [0,T]$ into N equidistant boundary elements $[t_{i-1}, t_i]$ for $i = \overline{1,N}$, $t_i = iT/N$ for $i = \overline{0,N}$, and the space interval $[0,l]$ into N_0 equidistant cells, $[x_{k-1}, x_k]$ for $k = \overline{1,N_0}$, $x_k = kl/N_0$ for $k = \overline{0,N_0}$.

Using a constant BEM we assume that the function u and its normal derivative $\frac{\partial u}{\partial n}$ are constant over each interval and we take the value at their midpoints $\tilde{t_i} = (t_i + t_{i-1})/2 = (2i-1)T/N$ for $i = \overline{1,N}$, and $\tilde{x_k} = (x_k + x_{k-1})/2 = (2k-1)l/N_0$ for $k = \overline{1,N_0}$, namely

$$u(0,t) = h_0(\tilde{t_i}) := h_{0i}, \quad u(l,t) = h_l(\tilde{t_i}) := h_{li} \tag{10}$$

$$\frac{\partial u}{\partial n}u(0,t) = \frac{\partial u}{\partial n}(0,\tilde{t_i}) := q_{0i}, \quad \frac{\partial u}{\partial n}u(l,t) = \frac{\partial u}{\partial n}(l,\tilde{t_i}) := q_{li} \tag{11}$$

$$u(x,0) = u_0(\tilde{x_k}) := u_{0k} \tag{12}$$

for $t \in [t_{i-1}, t_i]$, $i = \overline{1,N}$ and $x \in [x_{k-1}, x_k)$, $k = \overline{1,N_0}$.

With these approximations, the integral eqn. (8) can be approximated as

$$\eta(x)u(x,t) = \sum_{j=1}^{N}[A_{0j}(x,t)q_{0j} + A_{lj}(x,t)q_{lj} - B_{0j}(x,t)q_{0j} - B_{lj}(x,t)f_{lj}] + \\ + \sum_{k=1}^{N_0} C_k(x,t)u_{0,k} + D(x,t) \tag{13}$$

where

$$A_{0j}(x,t) = \int_{t_{j-1}}^{t_j} G(x,t,0,\tau)d\tau, \quad A_{lj}(x,t) = \int_{t_{j-1}}^{t_j} G(x,t,l,\tau)d\tau$$

$$B_{0j}(x,t) = \int_{t_{j-1}}^{t_j} \frac{\partial G}{\partial n(0)}(x,t,0,\tau)d\tau, \quad B_{lj}(x,t) = \int_{t_{j-1}}^{t_j} \frac{\partial G}{\partial n(l)}(x,t,l,\tau)d\tau$$

$$C_k(x,t) = \int_{x_{k-1}}^{x_k} G(x,t,y,0)dy, \quad \text{for } j = \overline{1,N}, \quad k = \overline{1,N_0} \tag{14}$$

$$D(x,t) = \int_0^l \int_0^t G(x,t,y,\tau)Q(y,\tau)d\tau dy \tag{15}$$

Special attention is considered now to the domain integral (15). We seek a piecewise constant approximation for the heat source $Q(y,\tau)$ and therefore we assume that

$$f(t) = f(\tilde{t_i}) := f_i, \quad t \in [t_{i-1}, t_i), \quad i = \overline{1,N} \tag{16}$$

The expression (15) is therefore approximated as

$$D(x,t) = \int_0^t f(\tau)\left(\int_0^l G(x,t,y,\tau)dy\right)d\tau = \sum_{j=1}^{N} D_j^I(x,t)f_j \tag{17}$$

where

$$D_j^I(x,t) = \int_{t_{j-1}}^{t_j}\int_0^l G(x,t,y,\tau)dyd\tau, \quad j = \overline{1,N} \tag{18}$$

$$\tag{19}$$

The integrals in eqns. (14) and (18) can be evaluated analytically and their expressions are given in the Appendix.

By applying the integral equation (13) at the boundary nodes $(0,\tilde{t_i})$ and $(l,\tilde{t_i})$ for $i = \overline{1,N}$, we obtain the system of $2N$ equations

$$\mathbf{Aq} - \mathbf{BF} + \mathbf{Cu_0} = 0 \tag{20}$$

where

$$A_{i,j} = A_{0j}(0,\tilde{t_i}), \quad A_{i,(j+N)} = A_{lj}(0,\tilde{t_i}), \quad i,j = \overline{1,N} \tag{21}$$

$$A_{(i+N),j} = A_{0j}(l,\tilde{t_i}), \quad A_{(i+N),(j+N)} = A_{lj}(l,\tilde{t_i}), \quad i,j = \overline{1,N} \tag{22}$$

$$B_{i,j} = B_{0j}(0,\tilde{t_i}) + 0.5\delta_{ij}, \quad B_{i,(j+N)} = B_{lj}(0,\tilde{t_i}), \quad i,j = \overline{1,N} \tag{23}$$

$$B_{(i+N),j} = B_{0j}(l,\tilde{t_i}), \quad B_{(i+N),(j+N)} = B_{lj}(l,\tilde{t_i}) + 0.5\delta_{ij}, \quad i,j = \overline{1,N} \tag{24}$$

$$q_j = q_{0j}, \quad q_{j+N} = q_{lj}, \quad F_j = h_{0j}, \quad F_{j+N} = h_{lj}, \quad j = \overline{1,N} \tag{25}$$

$$C_{i,k} = C_k(0,\tilde{t_i}), \quad C_{(i+N),k} = C_k(l,\tilde{t_i}), \quad k = \overline{1,N_0}, \quad i = \overline{1,N} \tag{26}$$

$$d_i = D(0,\tilde{t_i}), \quad d_{i+N} = D(l,\tilde{t_i}), \quad k = \overline{1,N_0}, \quad i = \overline{1,N} \tag{27}$$

where δ_{ij} is the Kronecker delta symbol.

Since the Neumann direct problem for the heat equation has a unique solution, the matrix $\mathbf{A}$ is invertible, and therefore from (20) we can eliminate the flux $\mathbf{q}$ to obtain

$$\mathbf{q} = \mathbf{A}^{-1}\mathbf{BF} - \mathbf{A}^{-1}\mathbf{Cu_0} - \mathbf{A}^{-1}\mathbf{d} \tag{28}$$

We now use the conditions (4) by applying the integral eqn. (8) at the point $(x_0, \tilde{t_i})$ for $i = \overline{1,N}$, to obtain the system of N equations

$$\chi = \mathbf{A^1 q} - \mathbf{B^1 F} + \mathbf{C^1 u_0} + \mathbf{d_1} \tag{29}$$

where

$$\chi_i = \chi(\tilde{t_i}), \quad i = \overline{1,N} \tag{30}$$
$$A^1_{i,j} = A_{0j}(x_0, \tilde{t_i}), \quad A^1_{i,(j+N)} = A_{lj}(x_0, \tilde{t_i}), \quad i,j = \overline{1,N} \tag{31}$$
$$B^1_{i,j} = B_{0j}(x_0, \tilde{t_i}), \quad B^1_{i,(j+N)} = B_{lj}(x_0, \tilde{t_i}), \quad i,j = \overline{1,N} \tag{32}$$
$$C^1_{i,k} = C_k(x_0, \tilde{t_i}), \quad d^1_i = D(x_0, \tilde{t_i}), \quad i = \overline{1,N} \quad k = \overline{1,N_0} \tag{33}$$
$$\tag{34}$$

The vectors $\mathbf{d}$ and $\mathbf{d_1}$ are given by

$$\mathbf{d} = \mathbf{D}^I \mathbf{f}, \qquad \mathbf{d_1} = \mathbf{D}^1 \mathbf{f} \tag{35}$$

where

$$D^I_{i,j} = D^I_j(0, \tilde{t_i}), \quad D^I_{(i+N),j} = D^I_j(l, \tilde{t_i}), \quad i,j = \overline{1,N} \tag{36}$$
$$D^1_{i,j} = D^I_j(x_0, \tilde{t_i}), \quad i,j = \overline{1,N} \tag{37}$$

Based on the above notations we obtain

$$(\mathbf{D}^1 - \mathbf{A}^1\mathbf{A}^{-1}\mathbf{D}^I)\mathbf{f} + (\mathbf{A}^1\mathbf{A}^{-1}\mathbf{B} - \mathbf{B}^1)\mathbf{F} + (\mathbf{C}^1 - \mathbf{A}^1\mathbf{A}^{-1}\mathbf{C})\mathbf{u_0} = \chi. \tag{38}$$

By introducing the following notation:

$$\mathbf{z_1} = \chi + (\mathbf{B}^1 - \mathbf{A}^1\mathbf{A}^{-1}\mathbf{B})\mathbf{F} + (\mathbf{A}^1\mathbf{A}^{-1}\mathbf{C} - \mathbf{C}^1)\mathbf{u_0} \tag{39}$$
$$\mathbf{X_1} = \mathbf{D}^1 - \mathbf{A}^1\mathbf{A}^{-1}\mathbf{D}^I \tag{40}$$

we obtain that the problem has been reduced to solving the following $N \times N$ system of linear equations:

$$\mathbf{X_1 f} = \mathbf{z_1} \tag{41}$$

From section 2 we know that this systems of linear equation has a unique solution, but it is ill-conditioned. Thus, if the right-hand side of eqn. (41)is contaminated with errors, i.e.

$$||\mathbf{z_1}^\epsilon - \mathbf{z_1}|| \leq \epsilon \tag{42}$$

then the inverted solution of (41), namely $\mathbf{X_1}^{-1}\mathbf{z_1}^\epsilon$ will be very different from the exact solution $\mathbf{X_1}^{-1}\mathbf{z_1}$. Therefore, instead of the straightforward inversion of (41) we employ the zeroth-order Tikhonov regularization method which gives the following solutions:

$$\mathbf{f}_\lambda = (\mathbf{X_1}^{tr}\mathbf{X_1} + \lambda\mathbf{I})^{-1}\mathbf{X_1}^{tr}\mathbf{z_1}^\epsilon \tag{43}$$

where λ is a regularization parameter which can be chosen according to the discrepancy principle, see [8], i.e. λ is choosen such that

$$||\mathbf{X_1 f}_\lambda - \mathbf{z_1}^\epsilon|| \approx \epsilon \tag{44}$$

Higher-order Tikhonov regularizations can also be employed, see [9].

4. NUMERICAL RESULTS AND DISCUSSION

In this section we present and discuss the numerical results obtained by employing the BEM combined with the Tikhonov regularization technique presented in Section 3, for a typical example. For this example we have taken $l = T = 1$ and $x_0 = 0.5$. The number of boundary elements was taken $N = N_0 = 40$, which was found to be sufficiently large to ensure that any further increase in this discretization did not significantly affect the accuracy of the numerical solutions of the direct problem (1)-(3) if $f(t)$ was known. The choice of the regularization parameters λ was based on the discrepancy principle.

With the input data

$$u(0,t) = h_0(t) = 2t + \cos(4\pi t), \quad u(1,t) = f_1(t) = 1 + 2t + \cos(4\pi t) \tag{45}$$
$$u(x,0) = u_0(x) = x^2, \quad u(0.5,t) = \chi(t) = 0.25 + 2t + \cos(4\pi t) \tag{46}$$

the inverse problem (1)-(4) has the unique solution given by

$$u(x,t) = x^2 + 2t + \cos(4\pi t), \tag{47}$$

$$f(t) = -4\pi \sin(4\pi t) \tag{48}$$

Figure 1 shows the numerical results obtained for estimating the time dependent heat source (48) when employing the Tikhonov zero-order regularization and when both exact and noisy data was used. It can be seen from Figure 1 that the numerical solution obtained in the ideal case when no noise is contained in the input data (45)-(46) is graphically almost indistinguishable from the analytical solution (48). In fact, the regularization parameter in this case was chosen to be $\lambda = 0$, which is equivalent to saying that no regularization is needed in this case and the numerical solution can be obtained using a standard Gaussian elimination technique.

Next, the input data (45)-(46) was perturbed by $p \in \{1, 3, 5\}$ percent random Gaussian additive noise. From the numerical results shown in Figure 1, it can be seen that the numerical solution is converging to the exact solution (48), as the amount of noise p decreases. Also it can be seen from Figure 1 that there are some inaccuracies in these numerical solutions which were obtained using zero-order Tikhonov regularization technique. The inaccuracies are clearly visible at the endpoints of the time interval, where the convergence of the numerical solution is particularly slow.

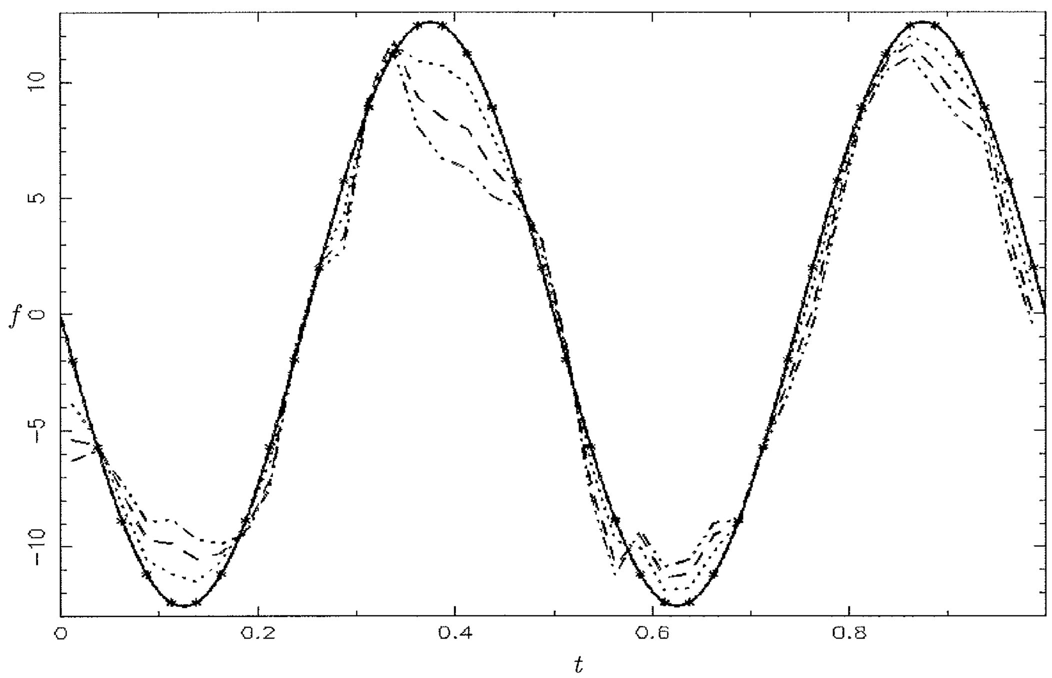

Figure 1: The numerical time-dependent heat source results obtained with exact data ($* \ * \ *$) and when using zero-order Tikhonov regularization for $p = 1$ ($\cdots$), $p = 3$ ($- \ - \ -$), and $p = 5$ ($- \cdots - \cdots -$) percent noise, in comparison with the exact solution (——).

Figure 2 shows the numerical results obtained using the same sets of noisy data as previously, but when first-order Tikhonov regularization was employed (in this case the identity matrix $\mathbf{I}$ in (43) should be replaced by an approximate matrix containing a first-order derivative of f, see [9]). The results are now clearly improved in accuracy when compared with the zero-order results of Figure 1, although some inaccuracies can be still seen towards the solution endpoints.

Finally, the most accurate results for this example were obtained when using second-order Tikhonov regularization and they are presented in Figure 3. The same amounts of random Gaussian noise as in the previous two cases were added to the input data (45)-(46), i.e. $p \in \{1, 3, 5\}$ percent. It can be seen from Figure 3 that the numerical solutions are all stable and very close to the analytical solution and that, as the amount of noise p decreases, they are converging to the exact solution (48).

The values of the regularization parameter λ were chosen according to the discrepancy principle and they are shown in Table 1.

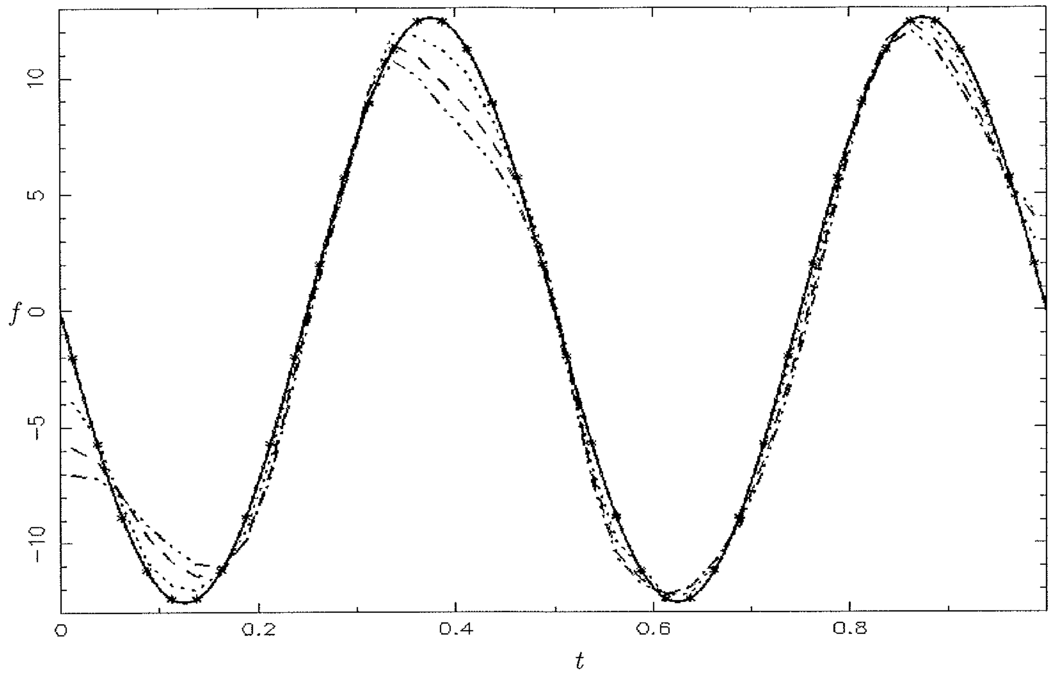

Figure 2: The numerical time-dependent heat source results obtained when using first-order Tikhonov regularization for $p = 1$ (···), $p = 3$ (− − −), and $p = 5$ (− · · · − · · · −) percent noise, in comparison with the exact solution (——).

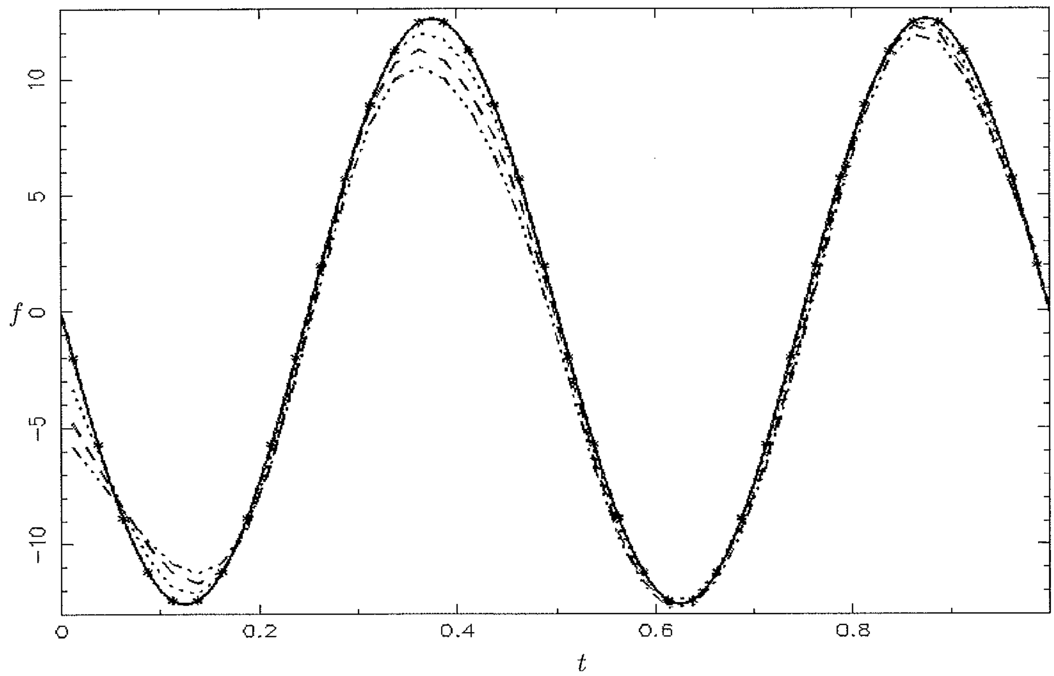

Figure 3: The numerical time-dependent heat source results obtained when using second order Tikhonov regularization for $p = 1$ (···), $p = 3$ (− − −), and $p = 5$ (− · · · − · · · −) percent noise, in comparison with the exact solution (——).

Table 1: The values of the regularization parameters λ.

Tikhonov order	$p = 1\%$	$p = 2\%$	$p = 3\%$
zero	0.00013	0.0004	0.00065
first	0.00050	0.0016	0.0025
second	0.0015	0.0079	0.0125

5. CONCLUSIONS

In this paper a BEM combined with a regularization technique has been developed for obtaining stable timewise dependent heat sources, from an over-specified condition which ensures the unique solvability for the inverse heat source problem. The numerical results obtained show that the BEM combined with Tiknonov regularization is capable of successfully recovering the heat source.

Although we only have considered Dirichlet boundary conditions, there are also solvability theorems for the problem of finding a source for the parabolic heat equation with general boundary conditions. A similar approach can be used for the determination of a single variable source for the heat conduction equation in another inverse formulation, with an integral overdetermination condition specifying the energy variation of the heat conducting system, see [5].

The study performed in this paper can be extended to higher dimensional parabolic partial differential equations of order n with constant coefficients $(a_i)_{i=\overline{1,n}}$, b and $(k_{ij})_{i,j=\overline{1,n}}$ positive definite, of the form

$$\sum_{i,j=1}^{n} k_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + bu = \frac{\partial u}{\partial t} + Q(\mathbf{x}, t) \tag{49}$$

where $\mathbf{x} = (x_1, ..., x_n) \in \Omega$, $t \in (0, T)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain and the unknown heat source $Q(\mathbf{x}, t)$ is independent of the space or time variable.

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Appendix 1

The discretization of the boundary integral eqn. (8) using the BEM results in expressions which involve integrals of G and $\partial G/\partial n$ and which are given by

$$A_{\xi j}(x,t) = \int_{t_{j-1}}^{t_j} G(x,t,\xi,\tau)d\tau$$

$$= \begin{cases} 0, & t \leq t_{j-1} \\ \sqrt{(t-t_{j-1})/\pi}, & t_{j-1} < t \leq t_j,\ r=0 \\ r(exp(-z^2)/z - \sqrt{\pi}erfc(z))/(2\sqrt{\pi}), & t_{j-1} < t \leq t_j,\ r \neq 0 \\ r\{\exp(-z^2)/z - \exp(-z_1^2)/z_1 & \\ +\sqrt{\pi}(\mathrm{erf}(z) - \mathrm{erf}(z_1))\}/(2\sqrt{\pi}), & t > t_j \end{cases} \quad (50)$$

$$B_{\xi j}(x,t) = \int_{t_{j-1}}^{t_j} \frac{\partial G}{\partial n(\xi)}(x,t,\xi,\tau)d\tau$$

$$= \begin{cases} 0, & t \leq t_{j-1} \\ 0, & t_{j-1} < t \leq t_j,\ r=0 \\ -\mathrm{erfc}(z)/2, & t_{j-1} < t \leq t_j,\ r \neq 0 \\ (\mathrm{erf}(z) - \mathrm{erf}(z_1))/2, & t > t_j \end{cases} \quad (51)$$

with $\xi \in \{0, l\}$, $r = | x - \xi |$, $z = r[(t-t_{j-1})]^{-1/2}/2$, $z_1 = r[(t-t_j)]^{-1/2}/2$,

$$C_k(x,t) = \int_{x_{k-1}}^{x_k} G(x,t,y,0)dy = \frac{1}{2}\left[\mathrm{erf}\left(\frac{x-x_{k-1}}{2\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x-x_k}{2\sqrt{t}}\right)\right] \quad (52)$$

$$D_j^I(x,t) = \int_{t_{j-1}}^{t_j}\int_0^l G(x,t,y,\tau)dyd\tau$$

$$= \begin{cases} 0, & t \leq t_{j-1} \\ -\frac{2x^2-2x+1}{4} - J_1(x,t,t_{j-1}), & t_{j-1} < t < t_j \\ J_1(x,t,t_j) - J_1(x,t,t_{j-1}), & t_j \leq t \end{cases} \quad (53)$$

$$J_1(x,t,t_0) = -\frac{r}{2}\mathrm{erf}(z) - \frac{1}{2\sqrt{\pi}}\frac{x\sqrt{r}}{\exp(z^2)} - \frac{x^2}{4}\mathrm{erf}(z)$$

$$+\frac{r}{2}\mathrm{erf}(z_1) + \frac{1}{\sqrt{\pi}}\frac{(x-l)\sqrt{r}}{2\exp(z^2)} + \frac{(x-l)^2}{4}\mathrm{erf}(z_1) \quad (54)$$

with $r = t - t_0$, $z = \frac{x}{2\sqrt{t-t_0}}$ and $z_1 = \frac{x-l}{2\sqrt{t-t_0}}$.
In the above expressions, $x \in [0,1]$, $t \in (0,1]$, and *erf* and *erfc* are the error functions defined as

$$\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\sigma^2}d\sigma, \quad \mathrm{erfc}(\xi) = 1 - \mathrm{erf}(\xi). \quad (55)$$